AdS/CFT Duality and Entanglement Entropy

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Introduction
Overview of the AdS/CFT theory

Ryu-Takayanagi Formula

Application

Let's start with the idea of Black Hole.

Black hole as a temperature parameter

	Thermodynamics	Black Hole
Zeroth Law	T is constant at equilibrium	Surface gravity κ is constant for a stationary solution
First Law	$dE = TdS$	$dM = \frac{\kappa}{8\pi G}dA$
Second Law	$dS \geq 0$	$dA \geq 0$
Third Law	$T \to 0 \Rightarrow S \to 0$	$T \to 0 \Rightarrow ?$

Table: No-hair law and basic principle suggests the thermodynamics-BH similarity

AdS space and Boundary

$$ds^2 = \frac{4\sum_{i=0}^{d} dy_i^2}{(1-|y|^2)^2}$$

For a closed boundary, for example, $\sum_{i=0}^{d} y_i^2 = 1$. Function $f = 1 - |y|^2$ is positive, and vanishes on boundary. Hence such a replacement $d\tilde{s}^2 = f^2 ds^2$would extend the matrix to boundary. Also, by $y = \tanh(y/2)$, one get

$$ds^2 = dy^2 + \sinh^2 y d\Omega^2, \ \ y \geq 0$$

where the boundary is $y = \infty$. Then transform more, one get

$$ds^2 = \frac{1}{z_0^2}\left(\sum_{i=0}^{d} dz_i^2\right)$$

where the boundary is $z_0 = 0$ and infty point $z_0 = \infty$.

AdS space and Boundary, cont

where the relationship between this matrix and the previous one is connected by conformal transformation

$$z \to \frac{z - i}{z + i}$$

Conformal Field Theory

Generators of symmetry: conformal invariant operations.

$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu),\ P_\mu = -i\partial_\mu,\ D = -ix_\mu\partial^\mu,\ K_\mu = i(x^2\partial_\mu - 2x_\mu x_\nu\partial^\nu)$$

Figure: $x^\mu \rightarrow \frac{x^\mu - a^\mu x^2}{1 - 2a\cdot x + a^2x^2}$ transformation

Duality: Dictionary

- Requirement: symmetries match.
- Partion function $Z_{CFT}[\phi] = Z_{AdS(Bulk)}[\phi]$
- A field whose boundary behavior is $z^{-\Delta}$ maps to an operator of dimension Δ:$\mathcal{O}(x) = \lim_{z\to 0} z^{-\Delta}\phi(x,z)$.
  $\langle\prod_n \mathcal{O}\rangle_{CFT} = \lim_{z\to 0} z^{-n\Delta}\langle\prod_n \phi(x_i,z)\rangle_{Bulk}$
- Maldacena (1997)
  - type IIB string theory corresponds to $N=4$ supersymmetric Yang–Mills theory on the four-dimensional boundary in the large N limit
- Strongly coupled $\rightarrow$ Weakly coupled $+$ Gravity!

Massless Scalar Field Examples (*)

Ryu-Takayanagi Formula

1-D quantum many-body entanglement entropy is given by

$$S_A = \frac{c}{3} \cdot \log\left(\frac{L}{\pi a} \sin\left(\frac{\pi l}{L}\right)\right)$$

where l and L are the length of the subsystem A and the total system $A \cup B$. Based on von Neumann entropy, define the entanglement entropy S_A in a CFT on $\mathbb{R}^{1,d}$ (or $\mathbb{R} \times S^d$) for a subsystem A that has an arbitrary $d-1$ dimensional boundary $\partial A \in \mathbb{R}^d$ (or S^d): 'area law'

$$S_A = \frac{\text{Area of } \gamma_A}{4G_N^{(d+2)}},$$

γ_A is the d dimensional static minimal surface in AdS_{d+2} whose boundary is given by ∂A

$$S = -\mathrm{Tr}\rho \ln \rho = -\left.\frac{\partial}{\partial n}\mathrm{Tr}_A \rho_A^n\right|_{n\to 1}, \& \ \frac{\partial \ln \mathrm{Tr}_A \rho_A^n}{\partial n} = \frac{1}{\mathrm{Tr}_A \rho_A^n}\frac{\partial \mathrm{Tr}_A \rho_A^n}{\partial n}$$

To connect n part, a conical defect is derived, which is mapped to a point mass at AdS(Bulk) space. Ricci scalar reads as

$$R = 4\pi(1-n)\delta(\gamma_A) + R^{(0)}$$

which then results in E-H entropy, i.e., $\ln Z_{\mathrm{CFT}}$. Use

$$S_{\text{E-H}} = -\frac{1}{16\pi G}\int d^3x(R+\Lambda) = -\frac{\text{Area}}{16\pi G}(1-n)\cdot 4\pi$$

and

$$S_A = -\frac{\partial}{\partial n}\ln \mathrm{Tr}_A \rho_A^n = -\frac{\partial}{\partial n}\left[\frac{(1-n)\text{Area}}{4G}\right]_{n=1} = \frac{\text{Area}}{4G}$$

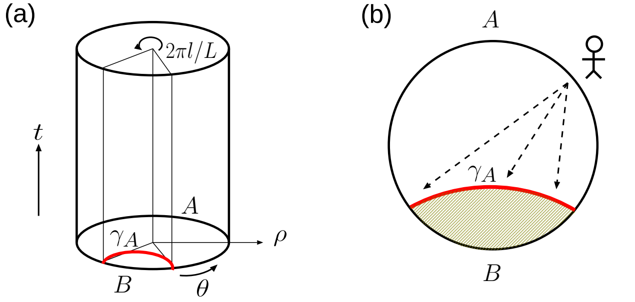


Figure: (a) AdS_3 space and CFT_2 living on its boundary and (b) a geodesics γ_A as a holographic screen.

It is readily seen that the basic properties of the entanglement entropy

- $S_A = S_B$ (B is the complement of A),
- $S_{A_1} + S_{A_2} \geq S_{A_1 \cup A_2}$ (subadditivity) and
- $S_{AB} + S_{BC} \geq S_B + S_{ABC}$ (strong subadditivity)

are satisfied.

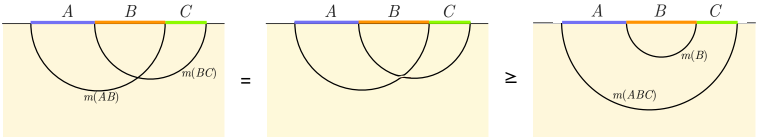


Figure: Strong subadditivity.

Finite Temperature and Black Hole

In this case, the time integral is then considered as a periodic one. As space is periodic already, we can get the time periodic as the space described by torus:

$$ds^2 = (r^2 - r_+^2)d\tau^2 + \frac{R^2}{r^2 - r_+^2}dr^2 + r^2 d\phi^2$$

while previously it's a cylinder ($t : -\infty \to \infty$). Then, the point is expanded due to finite time period, and relates by $\beta/L = R/r_+$, exactly a Euclidean black hole.
Recall the B-H and thermodynamics.

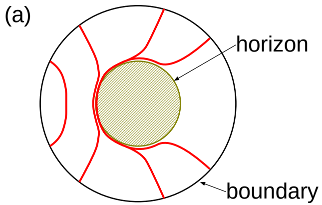
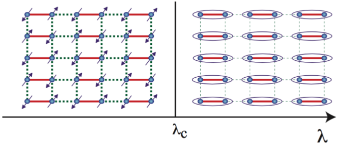


Figure: (a) Minimal surfaces γ_A for various sizes of A. (b) γ_A and γ_B wrap the different parts of the horizon.

Application: to CMT (Antiferromagnets)

Use effective field theory to describe entanglement entropy, use AdS/CFT to map to a known result, so that the exact solution and excitation spectrum is derived.

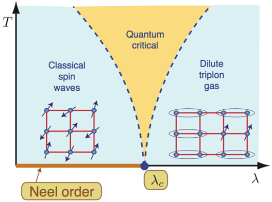


Order Parameter:
Staggered Polarization

$$\mathscr{S}_{LG} = \int d^d r d\tau \left[\frac{1}{2}\left[(\partial_\tau\varphi^a)^2 + v^2(\nabla\varphi^a)^2 + s(\varphi^a)^2\right] + \frac{u}{4}\left[(\varphi^a)^2\right]^2\right]$$

$$\varphi^a = z^*_\alpha \sigma^a_{\alpha\beta} z_\beta$$

$$\mathscr{S}_z = \int d^2 r d\tau \left[|(\partial_\mu - iA_\mu)z_\alpha|^2 + s|z_\alpha|^2 + u(|z_\alpha|^2)^2 + \frac{1}{2g^2}(\varepsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2\right]$$



➔ Extend to SCFT3
➔ Exact Solutions via AdS/CFT in large N limit!