Quantum Spin Hall Effect: a theoretical and experimental introduction at kindergarten level, non-shown version

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Historical consideration and Overview
Where everything get started

Theoretical: a picture
Preliminary
Graphene

Famous Experiments
Quantum Spin Hall Insulator State in HgTe Quantum Wells

Supplementary Material
Spin Hall Effect?

Why Spin Hall Effect

- Dissipationless quantum transport (unless TRS broken)[5] (cited by 1,531 times.)
Berry Phase: a quick introduction

Here\textsuperscript{1}, we consider a adiabatic dynamical evolution under a parametric Hamiltonian $\mathcal{H}(\mathbf{R})$ on its $n$-th eigenstate, for a closed loop in parameter space:

$$|n(\mathbf{R}(t))\rangle = e^{i\theta} |n(\mathbf{R}(0))\rangle,$$

where the additional phase is partially due to dynamical term,

$$\theta = \frac{1}{\hbar} \int_{0}^{t} E_{n}(\mathbf{R}(t')) dt' - i \int_{0}^{t} \left\langle n(\mathbf{R}(t')) \left| \frac{d}{dt'} \right| n(\mathbf{R}(t')) \right\rangle$$

An additional topological term comes from the second term if the system is topological nontrivial. A straight forward way of such nontrivial Hamiltonian is $\mathcal{H}(\mathbf{R}) = \epsilon(\mathbf{R}) + \mathbf{R} \cdot \sigma$ for a 2-state system.

\textsuperscript{1}For detailed information plz check my note of B.P via this link
Chern Number is also derived based on this. First we can define a vector (Berry Curvature) $V_n$ for its $i$ component is

$$V_{ni} = \text{Im} \sum_{m \neq n} \frac{\langle n(R)| (\nabla_R H(R))| m(R) \rangle \times \langle m(R)| (\nabla_R H(R))| n(R) \rangle}{(E_m(R) - E_n(R))^2}$$

and the nontrivial term, is

$$\gamma_n = - \int \int_c dS \cdot V_n$$

Chern number is simply integration over a closed surface of such vector and divided by $2\pi$. Note the relationship between Chern Number and Hall Conductance: $\sigma = \frac{e^2}{h} \cdot CN$. A good literature of this topic can be seen at [6].
Figure: TR protected impurity scattering with additional geometry phase $2\pi$ and interference cancel it (for spin, $V_\pm = \pm \frac{R}{2R^3}$, which gives a circle rotation of $\Omega = 2\pi$ yields $\gamma = \pi$, i.e., $e^{-i\gamma} = -1$)
Considering the work we have illustrated so far, the graphene is a good test platform. We have Kane-Mele Model for half-infinite case and yields edge state with chirality:
\[ E = -A \sin(k_x) \]
where \( s \) stands for spin\(^2\). It’s solvable and pointed out the requirement for the parameter.

\(^2\)details see further section
Quantum spin Hall effect in graphene (Haldane, Kane & Mele)

- SO coupling opens up a gap at the Dirac point.
- One pair of TR edge state on each edge.
- Numerical calculation indicate stability (Sheng et al)

Figure: From S-C Zhang’s Slides at 2005
A hint to solve (the Haldane model)

If you insist to know how to solve, there is a brief description and detailed information can be seen in my note as mentioned before.

1. Write down Hamiltonian, (for the sprite, we can suppose a simple cubic lattice). With fixed $k_z$, expand via $\Gamma$ algebra and yield bulk energy $E = \pm \sqrt{M^2(k) + A^2(\sin^2 k_x + \sin^2 k_y)}$

2. Discrete for $y$ direction is finite size and make fourier trans. $c_{k_x,k_y} = \frac{1}{L} \sum_j e^{ik_yj}c_{k_x,j}$ and write down Hamiltonian within this framework.

3. Use ansatz $\psi(k_x,j) = \lambda^{-j} \phi(k_x)$ and yield two eigenstate problem which is simply $4 \times 4$ case.
However graphene is disappointing, for its poor o-s coupling strength (Carbon is too light). All the hall phenomenon in graphene is summarized in YuanBo-Zhang’s [7] (cited by 8,872 times.).
Now, we consider the famous experiment: S-C Zhang et al: Quantum spin hall insulator state in HgTe quantum wells [8](cited by 2,594 times).
Close to fermi level, there are 4 bands.
CdTe vs. HgTe, w/ or w/o s-o coupling\textsuperscript{3}

\textit{s-like }$E_1$ band lies above \textit{p-like }$H_1$ band. Normal semiconductor.

\textsuperscript{3}Generated by MATLAB. Feel free to use my code. Using the Slater-Koster tight-binding method
CdTe vs. HgTe, w/ or w/o s-o coupling, cont.\textsuperscript{3}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{band_structure.png}
\caption{Band structure of HgTe with spin-orbit interaction.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{band_structure_no_so.png}
\caption{Band structure of HgTe without spin-orbit interaction.}
\end{figure}

$s$-like $E_1$ band lies behind $p$-like $H_1$ band. Inverted semiconductor.

\textsuperscript{3}Generated by MATLAB. Feel free to use my code. Using the Slater-Koster tight-binding method.
A ‘sandwich’ QW might influence the ‘inverty’ of the HgTe band if thickness is not enough. We then can make a mathematical description based on taylor expansion near Dirac point of our system.
BH$\bar{Z}$ model (by symmetry arguments, $\Gamma_6$ odd, $\Gamma_8$ even). Basis are $|E^{\pm}\rangle^4$, $|H^{\pm}\rangle^5$. Just suit Haldane model well.

$$H_{\text{eff}}(k_x, k_y) = \begin{pmatrix} H(k) & 0 \\ 0 & H^*(-k) \end{pmatrix},$$

$$H(k) = \epsilon(k) + \mathbf{d}(k) \cdot \sigma$$

\[4|E^+\rangle: \text{s-o state } |s, \uparrow\rangle; \quad |E^-\rangle: \quad |s, \downarrow\rangle; \quad \Rightarrow \quad |J = 1/2, m = \pm 1/2\rangle\]

\[5|H^+\rangle: \text{s-o state } |p_x + ip_y, \uparrow\rangle; \quad |H^-\rangle: \quad |p_x - ip_y, \downarrow\rangle; \quad \Rightarrow \quad |J = 3/2, m = \pm 3/2\rangle\]
With analyzing and the material’s property, we have really important information that

\[ d_3 = M - B(k_x^2 + k_y^2), \quad M : \text{ the mass parameter in Dirac-description} \]

CdTe: \( M < 0 \), HgTe: \( M \) might \( > 0 \).

**Figure:** Energy Band for the connecting CdTe and HgTe. For \( d_{QW} > d_c \) the HgTe layer becomes quantum spin Hall insulator. Massless helical states are confined on the sample edge. The sample has a finite conductance even when the Fermi level lies inside the bulk insulating gap.
Hence if make a Quantum Well with width has a critical point \( d_c \) where if \( d < d_c \), \( M < 0 \) and if \( d > d_c \), \( M > 0 \) (for the influence of the ‘sandwich cover’. Simu with 8 bands \( \mathbf{k} \cdot \mathbf{p} \) model of Hatree calculation [9](cited by 2,665 times.) shows that \( d_c = 6.3 \text{ nm} \) and confirmed by experiment.
A landau level argument

$|J = 3/2, m = -3/2\rangle$ will earn negative energy from LL, while $|J = 1/2, m = 1/2\rangle$ will earn positive energy from LL. If $d > d_c$, for a particular $B = B_c$ the two subband meet. This give rises to a strong magnetic field recover of Hall conductance $\sigma_{xy} = 0 \pm e^2/h$, + for conductance band and − for valence band. Higher field will then cancel it again and recover to $\sigma_{xy} = 0$. But this is hard to test from experiment.
Things to be measured: Longitude Resistance $R_{xx}$
Things to be measured: Four-term $R_{ij,kl}$

Landauer-Büttiker formalism, only valid for both-side\textsuperscript{6} edge-state transport.

$$I_i = \frac{e}{h} \sum_j (T_{ij}\mu_i - T_{ji}\mu_j)$$

$$T_{i,i+1} = T_{i+1,1} = 1$$

$$I_1 = -I_4 \equiv I_{14}$$

$$\mu_4 = 0 \quad \text{zero-point}$$

\[\begin{pmatrix}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_5 \\
\mu_6 \\
\end{pmatrix} = \frac{I_{14} \hbar}{e} \begin{pmatrix}
-3/2 \\
-1 \\
-1/2 \\
-1/2 \\
-1 \\
\end{pmatrix}\]

\textsuperscript{6}Simple Quantum Hall Effect doesn’t have $T_{i+1,i} = 1$
<table>
<thead>
<tr>
<th>$R_{ij,kl}$</th>
<th>Expected value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{14,14}$</td>
<td>$3h/2e^2$</td>
</tr>
<tr>
<td>$R_{14,23}$</td>
<td>$h/2e^2$</td>
</tr>
<tr>
<td>$R_{13,13}$</td>
<td>$4h/3e^2$</td>
</tr>
<tr>
<td>$R_{13,56}$</td>
<td>$h/3e^2$</td>
</tr>
</tbody>
</table>

**Figure:** Four-term $R_{ij,kl}$ measurement with precise gating
Experiment Result

Figure: (a) Hall resistance, different fermi level via gate voltage (b) Fermi level versus landau energy
Further

Spin accumulation is still a challenging task for experiment at that time. Eventually in 2012, German Molenkamp group (Würzburg uni) achieved it [10](ited by 118 times.). Theory suggested by Stanford plus Würzburg. Time limit and not gonna to be talked here.

Also, another subtle issue. It’s actually only ‘spin-like’ Hall effect for its not spin at all but the Kramer pair (due to the strong mix by s-o coupling). Also note the argument of relationship with $\mathbb{Z}_2$. In a topologically non-trivial system there must be odd number of Kramers’ pairs crossing the Fermi energy. A general review can be seen at the annual review [11].
M.I. Dyakonov and V.I. Perel.
Current-induced spin orientation of electrons in semiconductors.

J. E. Hirsch.
Spin hall effect.

Jairo Sinova, Dimitrie Culcer, Q Niu, NA Sinitsyn, T Jungwirth, and AH MacDonald.
Universal intrinsic spin hall effect.

*Spin Hall Insulator*, volume 93.


